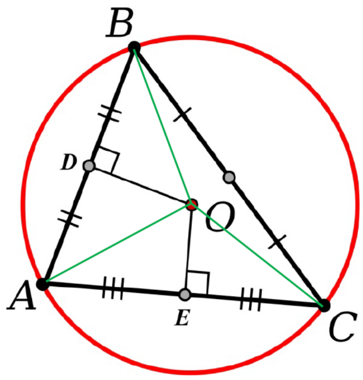
GEOMETRY ON THE PLANE (2)
CIRCUMCIRCLES AND INCIRCLES OF TRIANGLES
THEORY & PROBLEMS

1. CIRCUMCIRCLE OF A TRIANGLE

Theorem 1 *All triangles are cyclic, i.e. every triangle has a circumscribed circle or circumcircle.*



Proof. ΔABC, construct the perpendicular bisectors of sides AB and AC. (Recall that a perpendicular bisector is a line that forms a right angle with one of the triangle's sides and intersects that side at its midpoint.) These bisectors will intersect at a point O. So we have that $OD \perp AB$ and $OE \perp AC$. Now observe that $\Delta ODA \cong \Delta ODB$ by the *Side-Angle-Side Theorem*.
Thus, $OA = OB$ being corresponding sides of congruent triangles. It is also the case that $\Delta OEA \cong \Delta OEC$ by the *Side-Angle-Side Theorem*. So $OA = OC$ being corresponding sides of congruent triangles. Now consider any point X on segment AE. We find from the Pythagorean Theorem that

$$OA = \sqrt{OE^2 + EA^2} = \sqrt{OE^2 + (EX + XA)^2} > \sqrt{OE^2 + EX^2} = OX.$$

Therefore, $OA > OX$. In a similar way, we can establish that $r = OA = OB = OC$ is greater than the distance from Oto any other point Z on ΔABC. Hence, the circle with center at O and radius r circumscribes the triangle.

The circumcenter's position depends on the type of triangle

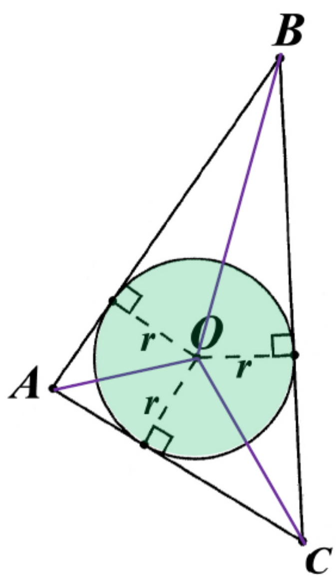
(a) If and only if a triangle is **acute** (all angles smaller than a right angle) the circumcenter lies inside the triangle.

(b) If and only if it is **obtuse** (has one angle bigger) the circumcenter lies outside the triangle.

(c) If and only if it is a **right** the circumcenter lies at the center of the hypotenuse.

2. INCIRCLE OF A TRIANGLE

Theorem 2 *A circle can be inscribed in any triangle, i.e. every triangle has an incircle.*



Proof. Given ΔABC, bisect the angles at the vertices A and B. These angle bisectors must intersect at a point, O . Locate the points D, E and F on sides AB, BC and CA respectively so that $OD \perp AB$, $OE \perp BC$ and $OF \perp CA$. Observe that $\Delta AOD \cong \Delta AOF$ and $\Delta BOD \cong \Delta BOE$ by the *Angle-Side-Angle Theorem*. Since corresponding sides of congruent triangles are equal, we also know that $OD = OF$ and $OD = OE$. Hence, $OD = OF = OE$. Moreover, it follows from the Pythagorean Theorem, that $r = OD = OF = OE$ is the shortest distance from the point O to each of the sides of ΔABC. So the circle with center O and radius r is an incircle for the triangle. Further, it is the only one, since any point equidistant from segments AB and BC must necessarily lie on line OB: the bisector of $\angle ABC$; and similarly, any point equidistant from segments CA and AB must lie on line OA: the bisector of $\angle CAB$. Therefore, O must be the center of the incircle for the ΔABC. Consequently, the incircle for any triangle is unique.

PROBLEMS

1. On sides BC, CA and AB of triangle ABC, points A_1, B_1 and C_1, respectively, are taken so that $AC_1 = AB_1$, $BA_1 = BC_1$ and $CA_1 = CB_1$. Prove that A_1, B_1 and C_1 are the points at which the inscribed circle is tangent to the sides of the triangle.

2. Let O_a, O_b and O_c be the centers of the escribed circles of triangle ABC. Prove that points A, B and C are the bases of heights of triangle $O_aO_bO_c$.

3. Prove that side BC of triangle ABC subtends:

   (1) an angle with the vertex at the center O of the inscribed circle; the value of the angle is equal to $90° + \frac{1}{2}|\angle A|$;

   (2) an angle with the vertex at the center O_a of the escribed circle; the value of the angle is equal to $90° - \frac{1}{2}|\angle A|$.

4. Inside triangle ABC, point P is taken such that

$$|\angle PAB| : |\angle PAC| = |\angle PCA| : |\angle PCB| = |\angle PBC| : |\angle PBA| = x.$$

   Prove that $x = 1$.

5. Let A_1, B_1 and C_1 be the projections of an inner point O of triangle ABC to the heights. Prove that if the lengths of segments $AA1$, BB_1 and CC_1 are equal, then they are equal to $2r$ (r the radious of the inscribed circle).

6. An angle of value $\alpha = |\angle BAC|$ is rotated about its vertex O, the midpoint of the basis AC of an isosceles triangle ABC. The legs of this angle meet the segments AB and BC at points P and Q, respectively. Prove that the perimeter of triangle PBQ remains constant under the rotation.

7. In a scalene triangle ABC, line MO is drawn through the midpoint M of side BC and the center O of the inscribed circle. Line MO intersects height AH at point E. Prove that $AE = r$.

8. A circle is tangent to the sides of an angle with vertex A at points P and Q. The distances from points P, Q and A to a tangent to this circle are equal to u, v and w, respectively. Prove that $\frac{uv}{w^2} = \sin^2\left(\frac{1}{2}\angle A\right)$.

9. Prove that the points symmetric to the intersection point of the heights of triangle ABC through its sides lie on the circumscribed circle.

10. From point P of arc BC of the circumscribed circle of triangle ABC perpendiculars PX, PY and PZ are dropped to BC, CA and AB, respectively. Prove that

$$\frac{|BC|}{|PX|} = \frac{|AC|}{|PY|} + \frac{|AB|}{|PZ|}.$$

11. Let O be the center of the circumscribed circle of triangle ABC, let I be the center of the inscribed circle, Ia the center of the escribed circle tangent to side BC. Prove that

    (a) $d^2 = R^2 - 2Rr$, where $d = OI$;

    (b) $d_a^2 = R^2 + 2Rr_a$, where $d_a = OI_a$.

12. The extensions of the bisectors of the angles of triangle ABC intersect the circumscribed circle at points A_1, B_1 and C_1; let M be the intersection point of bisectors. Prove that

(a) $\frac{|MA| \cdot |MC|}{|MB_1|} = 2r$;

(b) $\frac{|MA_1| \cdot |MC_1|}{|MB|} = R$.

13. The lengths of the sides of triangle ABC form an arithmetic progression: a, b, c, where $a < b < c$. The bisector of angle $\angle B$ intersects the circumscribed circle at point B_1. Prove that the center O of the inscribed circle divides segment BB_1 in halves.

14. In triangle ABC side BC is the shortest one. On rays BA and CA, segments BD and CE, respectively, each equal to BC, are marked. Prove that the radius of the circumscribed circle of triangle ADE is equal to the distance between the centers of the inscribed and circumscribed circles of triangle ABC.

SOURCES:

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